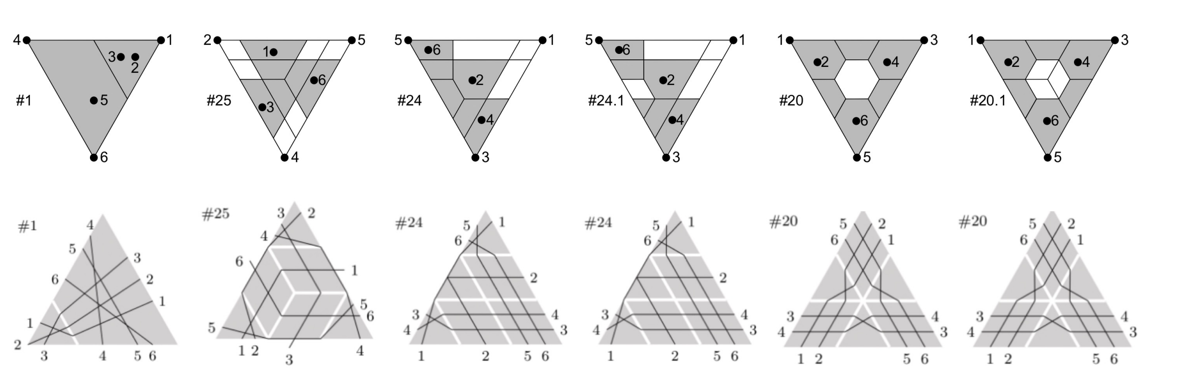
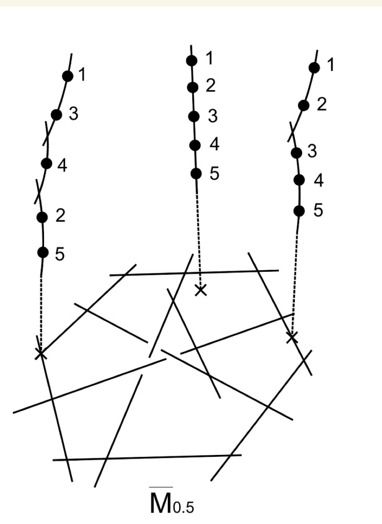
Compact moduli of points & lines in $\mathbb{P}^2$

(Jenia Tevelev, UMass Amherst)



joint w. Luca Schaffler, https://arxiv.org/abs/2010.03519

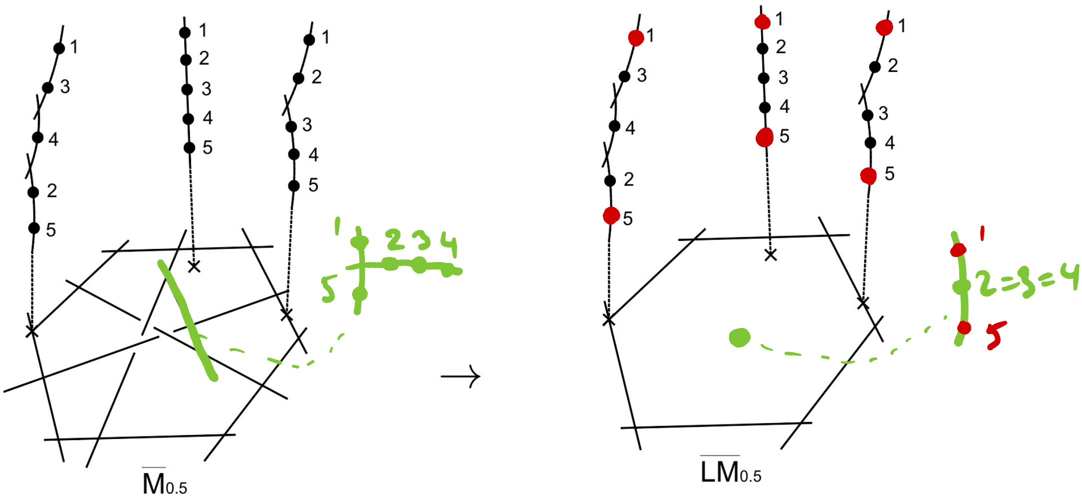
Compact moduli of points in $\mathbb{P}^1$: $M_{0,n} \hookrightarrow \overline{M}_{0,n}$



- $\overline{M}_{0,n}$ is **modular**: carries a family of stable rational curves
- **One-parameter limits** can be computed algorithmically (Mumford's semi-stable reduction or Kapranov's Bruhat-Tits tree method)
- $M_{0,n}$ is very affine and schön, $\overline{M}_{0,n}$ is a **tropical compactification** with a normal crossing boundary encoded in $\mathrm{Trop}(M_{0,n})$, a "tropical moduli space" of philogenetic trees

- $\overline{M}_{0,n}$ is a member of the family of Hassett's moduli spaces (weighted points) incl. the Losev-Manin space (2 heavy + light points)

trees of $\mathbb{P}^1$'s   chains of $\mathbb{P}^1$'s



tropical compactification   toric variety

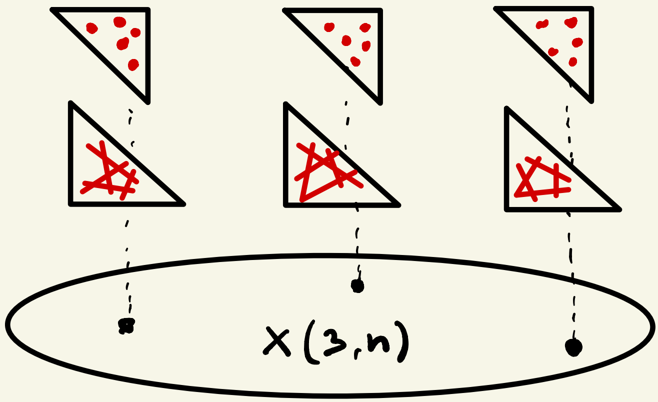
- Cross-ratios $\quad M_{0,n} \to \mathbb{P}^1$

$$(p_1 \dots p_n) \longmapsto \frac{p_1-p_3}{p_1-p_4} : \frac{p_2-p_3}{p_2-p_4}$$

extend to $\overline{M}_{0,n}$ and give an embedding $\overline{M}_{0,n} \hookrightarrow (\mathbb{P}^1)^{\binom{n}{4}}$ (Hacking-Keel-T)

Compact moduli of points and lines in $\mathbb{P}^2$

Are there compactifications with modularity, 1-parameter limits, tropicality, weights, cross-ratios for $X(3,n)$, the moduli space of



n points in $\mathbb{P}^2$,

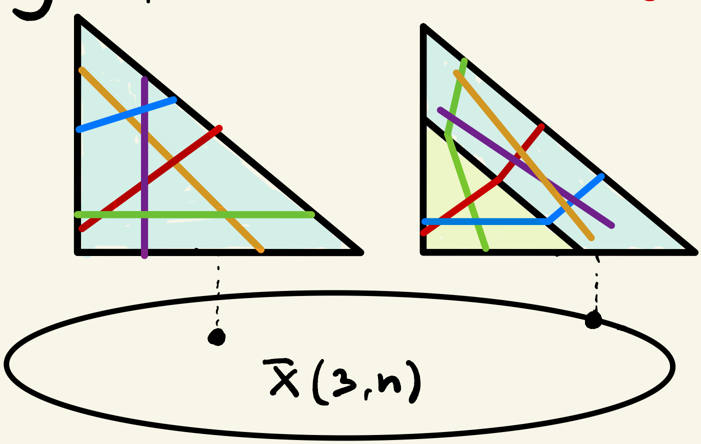
$\Updownarrow$

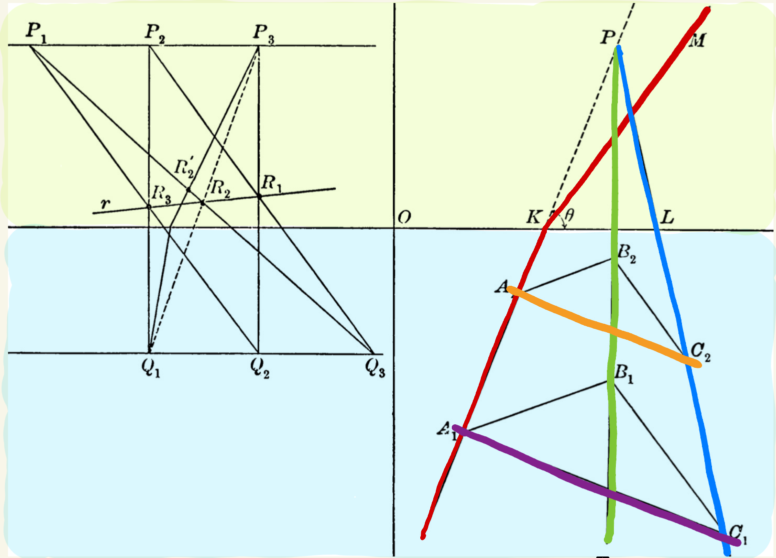
n lines in $(\mathbb{P}^2)^*$

in linearly general position

Kapranov (92), Chow Quotients of Grassmannians, I

- Compactification $X(3,n) \hookrightarrow \overline{X}(3,n)$
- A family of visible contours





An amuzing example of a "broken plane":
F. Moulton (1902), "A simple non-Desarguesian plane geometry", Transactions of AMS 3(2), 192–195

Kapranov raised various questions "to be addressed in part II" but instead part II was written collectively in 2000's.

- Modularity (Hacking–Keel–T): $\bar{X}(3,n)$ (with Kapranov's family) is an example of the Kollár–Shepherd-Barron–Alexeev moduli space of stable surfaces

- One-parameter limits: from modularity: semi-stable reduction + relative MMP.
  Also (Keel–T), Bruhat–Tits building method

• **Tropicality:** $X(3,n)$ is very affine and $\overline{X}(3,n)$ is the closure of $X(3,n)$ in the toric variety given by the Dressian fan (Kapranov)
$\Rightarrow \overline{X}(3,n)$ inherits combinatorics of matroid decompositions of the hypersimplex $\Delta(3,n)$ (Lafforgue): compactified configuration spaces of non-generic hyperplane arrangements stratified by matroid decompositions of their matroid polytope

$X(3,6)$ (Luxton) and $X(3,7)$ (Corey) are schön and $\overline{X}(3,6)$, $\overline{X}(3,7)$ are tropical compactifications

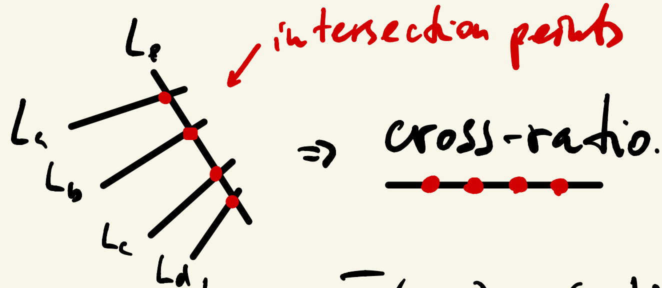
Typically, the Dressian is bigger than trop $X(3,n)$ (tropical Grassmannian of Sturmfels-Speyer) $\Rightarrow \overline{X}(3,n)$ is not a tropical compactification but it is "close" to it.

trop $X(3,n)$ is a tropical moduli space of tropical planes (Sturmfels-Speyer)

• **Weights:** (Alexeev) A weighted analogue of $\overline{X}(3,n)$, including a toric space $\overline{X}_{LM}(3,n)$

with the family of broken toric varieties with three heavy torus-invariant broken lines and $n-3$ broken light lines.

- **Cross-ratios**



(Kapranov) give a morphism $\overline{X}(3,n) \to (\mathbb{P}^1)^{n\binom{n-1}{4}}$
(Luxton) finite & birational onto its image $\mathbb{B}_n$

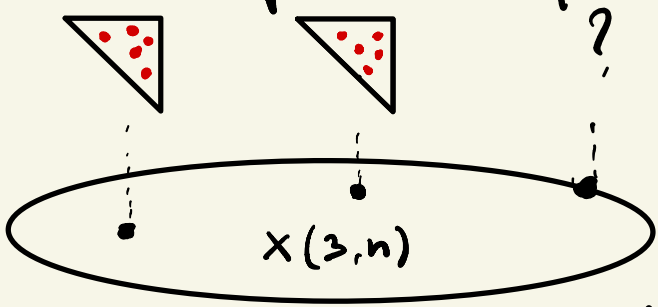
It is a closed embedding for $n=6$ (Luxton)

(Schaffler-T) The cross-ratio morphism is finite for all Lafforgue's compactified configuration spaces. $r \geq 3$ or $r=2, n\geq 5$

Matroid lemma Let $P, P' \subset \Delta(r,n)$ be full-dimensional matroid polytopes such that $P \subsetneq P'$. Then there exists a facet $E \subset \Delta(r,n)$ such that $P|_E, P'|_E$ are full-dimensional and $P|_E \subsetneq P'|_E$.

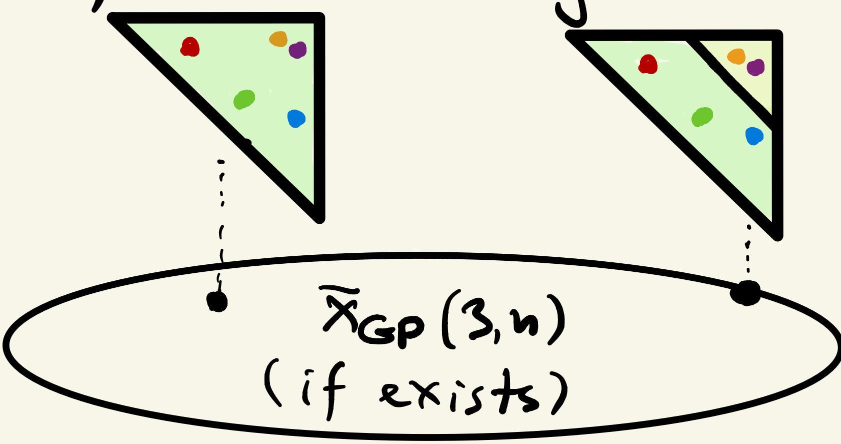
Gerritzen and Piwek (91)

Compact moduli space of n points in $\mathbb{P}^2$?,



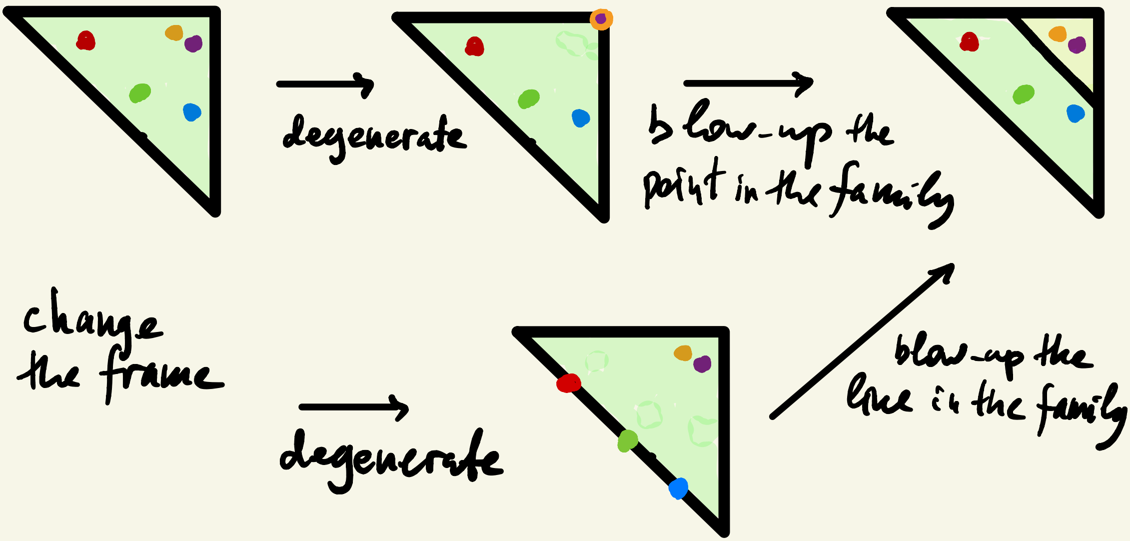
No KSBA-style moduli of points to guide us but some reasonable requirements on the universal family with n sections:

- special fibers reduced, Cohen-Macaulay
- n points $p_1 \dots p_n$ should be smooth
- Points should be disjoint or (weighted version) collide according to weights



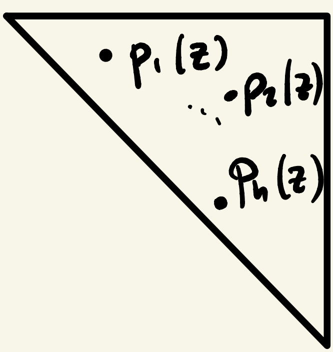
How is this different from the Fulton-Macpherson compactified configuration space of n points in $\mathbb{P}^2$?

In their case $\mathbb{P}^2$ is fixed and points move:



not degenerate from the FM perspective but it is focus

Gerritzen & Piwek start by identifying one-parameter limits



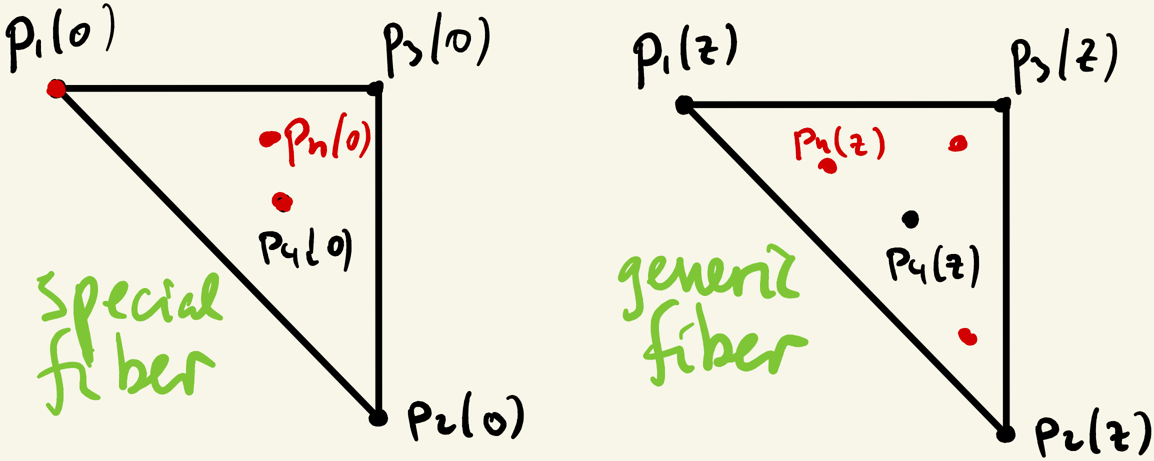
$$p_i(z) \in \mathbb{P}^2(R)$$

$$R = \mathbb{C}[(z)]$$

Choose a fourtuple $I \subset \{1,2,\dots,n\}$, for example $\{1,2,3,4\}$ and kill PGL_3-action

$p_1 \to [1:0:0]$, $p_2 \to [0:1:0]$, $p_3 \to [0:0:1]$, $p_4 \to [1:1:1]$

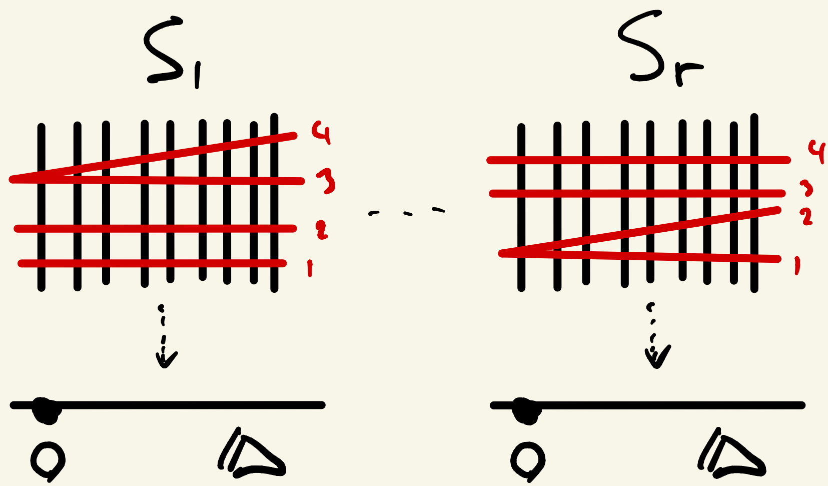
This gives a one-parameter family over Spec R



Points p_1, p_2, p_3, p_4 stay in general position but other points can go anywhere!

How to eliminate dependence of the limit on a fourtuple?

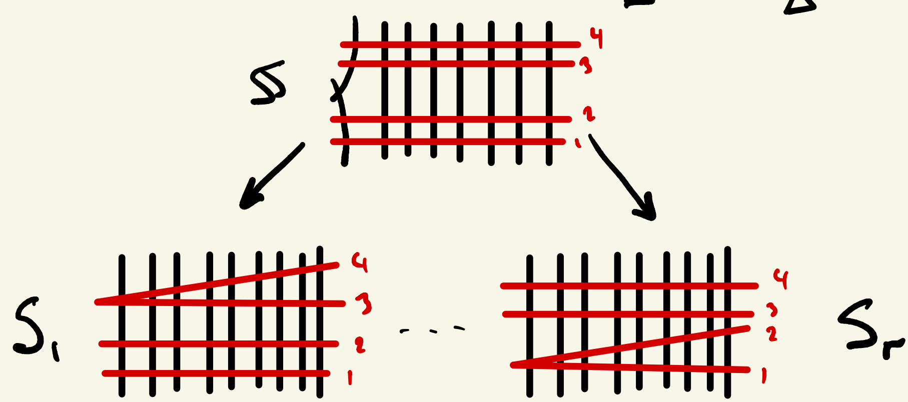
Mustafin join of bundles: given r families over the disc isomorphic over a punctured curve (or disc)



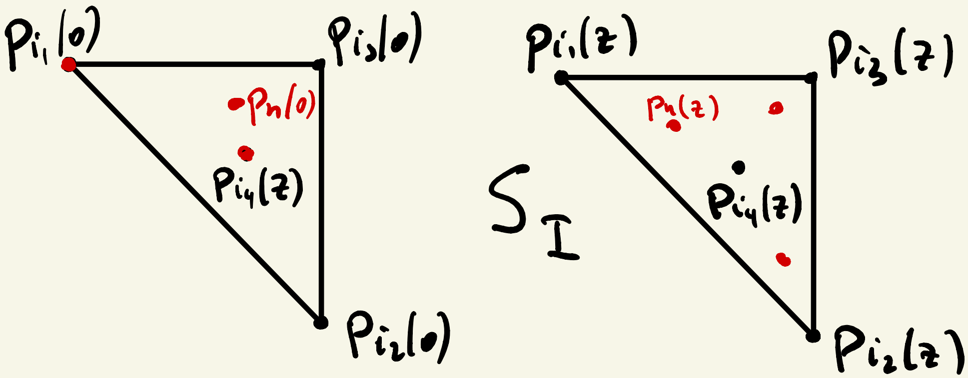
$$\pi_1^{-1}(\Delta^*) \simeq \dots \simeq \pi_r^{-1}(\Delta^*)$$

Embed $\pi^{-1}(\Delta^*) \hookrightarrow S_1 \times_\Delta \dots \times_\Delta S_r$

$$\mathbb{S} := \overline{\pi^{-1}(\Delta^*)} \hookrightarrow S_1 \times_\Delta \dots \times_\Delta S_r$$



Def For an arc $p_1(z), \dots, p_n(z) \in \mathbb{P}^2(R)$, the Gerritzen-Piwek family $\mathcal{S}$ over $\operatorname{Spec} R$ is the Mustafin join of $\binom{n}{4}$ $\mathbb{P}^2$-bundles S_I for every fourtuple $I \subset \{1, \dots, n\}$

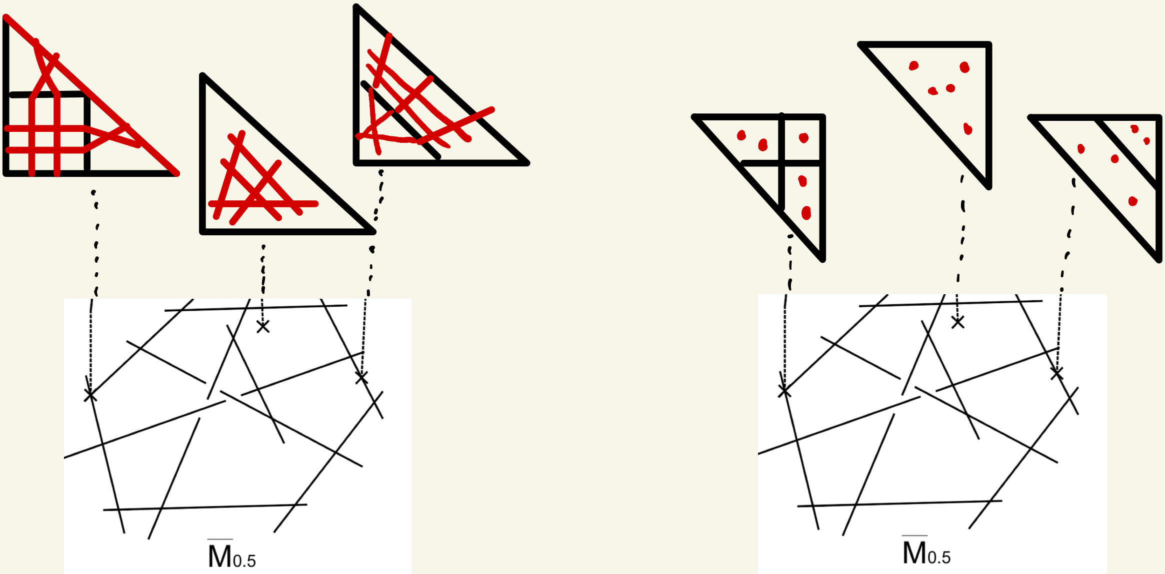


(Cartwright-Sturmfels) The special fiber $\mathcal{S}_0$ is reduced, Cohen-Macaulay

(Schaffler-T) $p_1, \dots, p_n \in \mathcal{S}_0$ are smooth disjoint points (claimed by Gerritzen-Piwek w/o proof)

Is there a moduli space $\overline{X}_{GP}(3,n)$ parametrizing special fibers $\mathcal{S}_0$ of arcs together with the universal family?

By hand: $\overline{X}(3,5) \simeq \overline{X}_{GP}(3,5) \simeq \overline{M}_{0,5}$



Kapranov asked ~2003 how to relate $\overline{X}(3,n)$ and $\overline{X}_{GP}(3,n)$.

Gerritzen and Piwek claimed that $\overline{X}_{GP}(3,n) \simeq \mathbb{B}_n$, the closure of $X(3,n)$ in $(\mathbb{P}^1)^{5\binom{n-1}{4}}$ defined by cross-ratios.

By Luxton theorem, $\overline{X}_{GP}(3,n)$ and $\overline{X}(3,n)$ would be the same up to normalization, ideal projective duality in families.

But this is wrong!

Gerritzen and Piwek define $\binom{n}{4}$ $\mathbb{P}^2$-bundles with n sections $S_I \to \mathbb{B}_n$ which all agree over $X(3,n) \hookrightarrow \mathbb{B}_n$.

$\Rightarrow$ universal Mustafin join

$$\mathcal{S} = \overline{\pi^{-1}(X(3,n))} \subset S_{I_1} \times_{\mathbb{B}_n} \cdots \times_{\mathbb{B}_n} S_{I_{\binom{n}{4}}}$$

But: $\mathbb{B}_n$ is not a smooth curve, so $\mathcal{S} \xrightarrow{\pi} \mathbb{B}_n$ is not automatically flat. In fact it is not even equidimensional.

Canonical solution: flattening by blow-up (Grothendieck-Raynaud-Gruson)

$$\begin{tikzcd} S = \mathcal{S}^{st} \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] \\ \overline{X}_{GP}(3,n) \arrow[r] & \mathbb{B}_n \end{tikzcd}$$

$\overline{X}_{GP}(3,n)$ is the closure of $X(3,n)$ in the Hilbert scheme of $\mathcal{S}$.

For technical reasons (e.g. to use Cartwright-Sturmfels) we use a multi-graded Hilbert scheme.

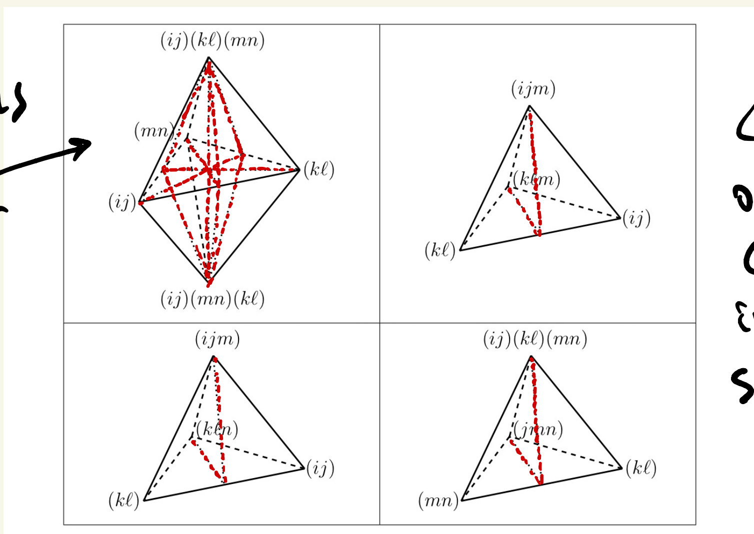
Th (Schaffler–T) • There exists $\overline{X}_{GP}(3,n)$ with a universal family S and n universal sections $s_1 \dots s_n : \overline{X}_{GP}(3,n) \to S$

• A morphism $\overline{X}_{GP}(3,n) \to \mathbb{B}_n \subset (\mathbb{P}^1)^{n\binom{n-1}{4}}$ extending cross-ratios exists but has fibers of positive dimension for $n \geq 6$

Example ($n=6$) $\overline{X}(3,6)$ is a tropical compactification given by trop $X(3,6)$ as described by **Sturmfels–Speyer**.

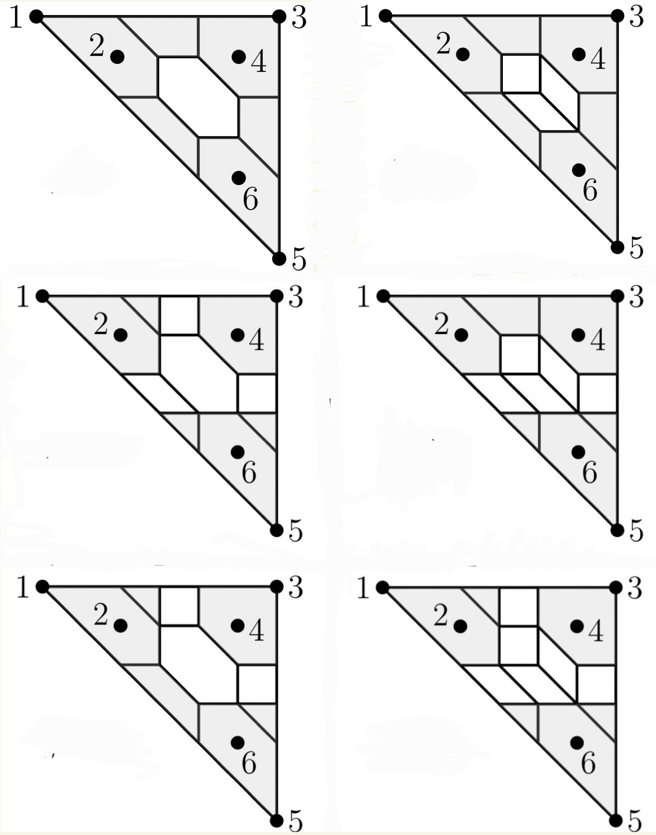
$\overline{X}_{GP}(3,6)$ is also tropical but it is given by the following subdivision:

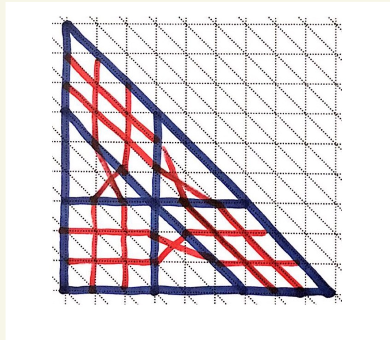
bipyramids
$\|$
15 singular points of $\overline{X}(3,6)$



4 types of maximal cones in trop $X(3,6)$ subdivided in $X_{GP}(3,6)$

Corresponding special fibers:

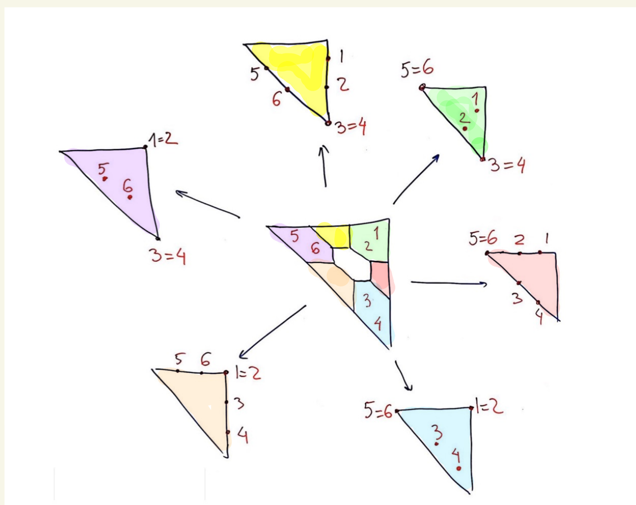




bi pyramid

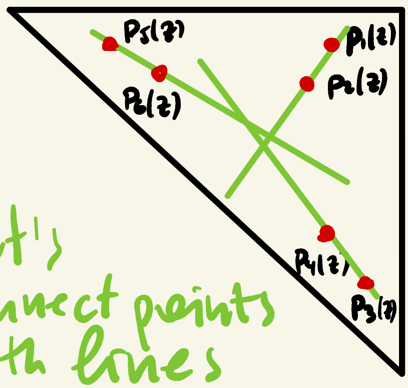
where do the extra cross-ratios come from?

Analysis of the Mustafin join



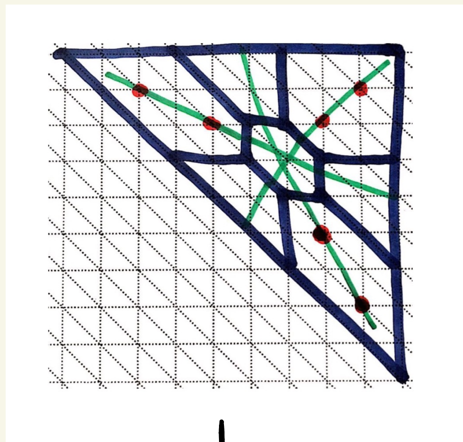
Each of the $\binom{6}{4}=15$ fourtuples is in general position in one of the $\mathbb{P}^2$s.

The central hexagon is a secondary component, it appears in the process of joining six $\mathbb{P}^2$-bundles

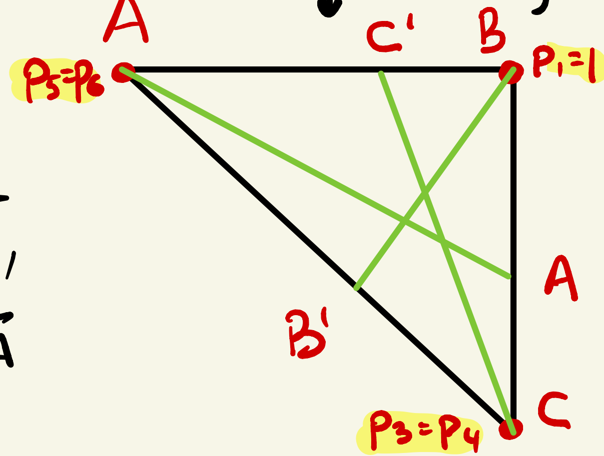


Let's connect points with lines

degenerate



contract onto hexagon



Čeva parameter

$$\mu = \frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A}$$

$\overline{X}_{GP}(3,6)$ (but not $\overline{X}(3,6)$!) is aware of

$\mu = 1 \Leftrightarrow AA', BB', CC'$ concurrent (Čeva)

$\mu = -1 \Leftrightarrow A', B', C'$ collinear (Menelaus)

Open Question: tropicality: is $\overline{X}_{GP}(3,n)$ the closure of $X(3,n)$ in some toric variety of the intrinsic torus? What is the analogue of the matroid decomposition?
How to describe $\overline{X}^{0}_{GP}(3,n) \xrightarrow{\pi} \overline{X}^{0}(3,n)$ combinatorially?

Partial answer for **the planar locus**

$$\mathcal{U}(3,n) = \left\{ x \in \overline{X}^{0}(3,n) : \begin{array}{l}\text{all matroid polytopes} \\ \text{in the matroid decomposition} \\ \text{contain some vertex} \\ e_i+e_j+e_k \text{ of } \Delta(3,n)\end{array} \right\}$$

$$= \bigcup \mathcal{U}^{ijk}(3,n) \qquad \text{open charts}$$

$$\mathcal{U}_{GP}(3,n) = \pi^{-1}(\mathcal{U}(3,n)) = \bigcup \mathcal{U}^{ijk}_{GP}(3,n)$$

All $\mathcal{U}^{ijk}$'s are isomorphic, so focus on $\mathcal{U}^{123}$

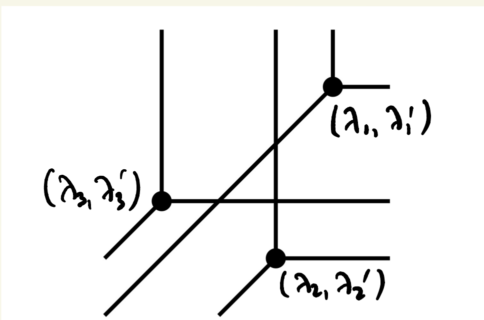
Th (Schaffler-T) The chart $\mathcal{U}^{123}$ (resp. $\mathcal{U}^{123}_{GP}$) is isomorphic to a (non-toric) open subset of a toric variety Q^{*} (resp. Q_{GP}). The map $Q_{GP} \to Q^{*}$ is toric and explicit.

In particular, the map $\mathcal{U}_{GP}(3,n) \to \mathcal{U}(3,n)$ will be easy to describe combinatorially.

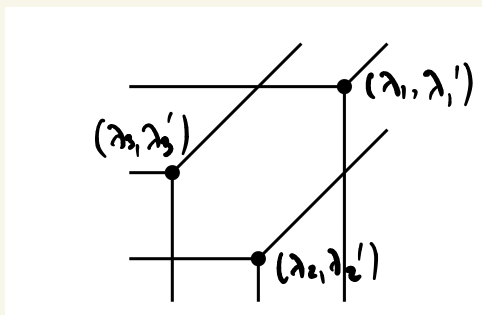
Def Q_m = normalized Chow quotient $(\mathbb{P}^2)^m // (\mathbb{C}^*)^2$

Q_m^* = - - - - - - - - - - - - - - - $(\mathbb{P}^{2})^{*m} // (\mathbb{C}^*)^2$

Q_m and Q_m^* are toric varieties of the quotient torus $(\mathbb{C}^*)^{2m}/(\mathbb{C}^*)^2$ given by the quotient fans: a 1-PS $\lambda \in \mathbb{Z}^{2n}/\mathbb{Z}^2$ can be encoded in a "spider diagram" modulo translations in $\mathbb{Z}_2$



Q_m

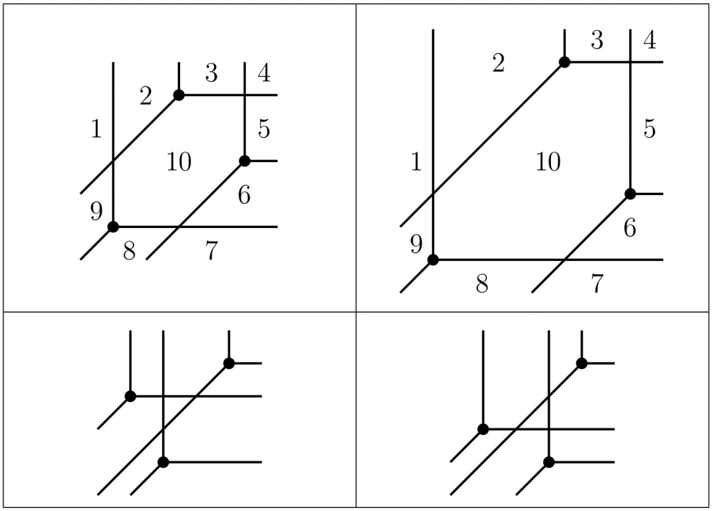


Q_m^*

Two one-parameter subgroups belong to the interior of the same cone in the quotient fan $\Leftrightarrow$ spider diagrams are combinatorially equivalent

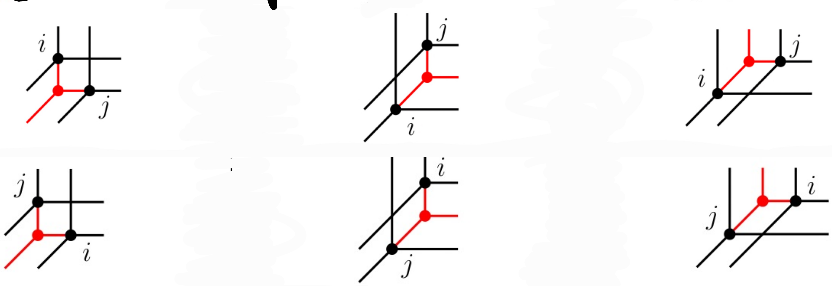
equivalent diagrams

not equivalent diagrams

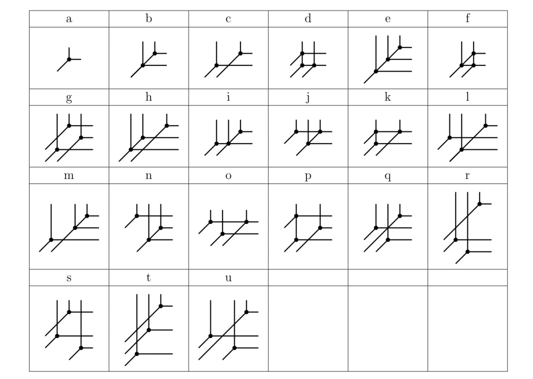


Next, we define a new fan $\widetilde{Q}_m$

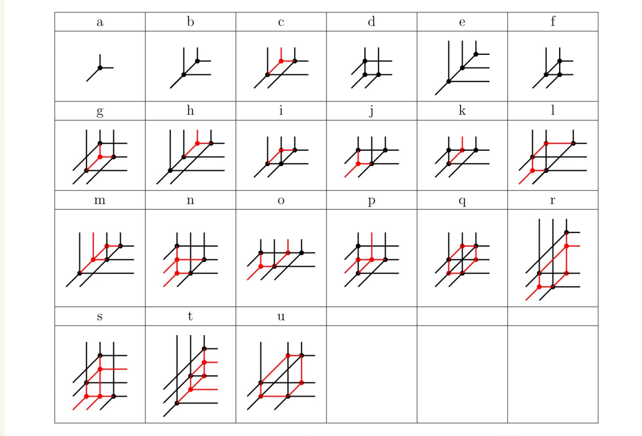
Adding red spiders to diagrams:



Do it for every pair of black dots:

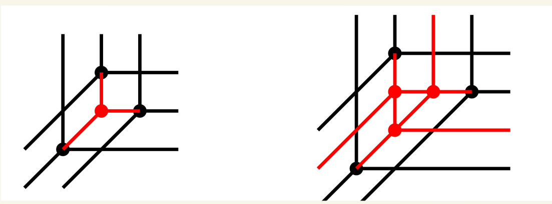


Cones in Q_3 (Mustafin triangles of Cartwright-Häbich-Sturmfels-Werner)



with added red spiders

This increases the number of combinatorial types:



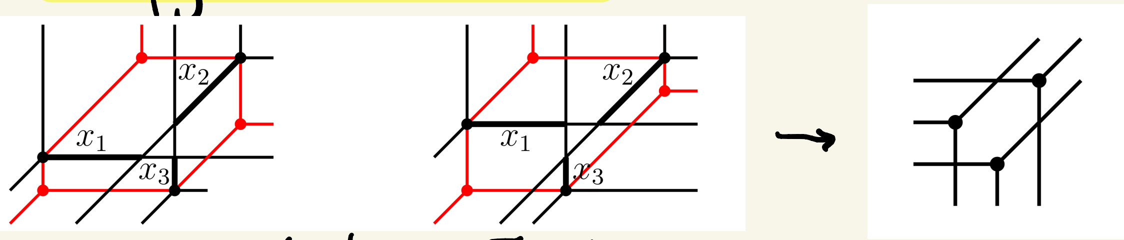
equivalent before adding red spiders but not after

Def The fan $\widetilde{Q}_m$ is defined as follows: two one-parameter subgroups are in the interior of the same cone $\Leftrightarrow$ diagrams with added red spiders are combinatorially equivalent

Lemma The fan $\widetilde{Q}_m$ refines both Q_m and Q_m^* giving birational morphisms of toric varieties

$$\begin{array}{ccccccc} & & X(\widetilde{Q}_m) & \supset & U^{123}_{GP}(3,n) & \subset & \overline{X}^0_{GP}(3,n) \\ & & \downarrow & & \downarrow & & \downarrow \\ \overline{X}_{LM}(3,n) & = & X(Q_m^*) & \supset & U^{123}(3,n) & \subset & \overline{X}^0(3,n) \end{array}$$

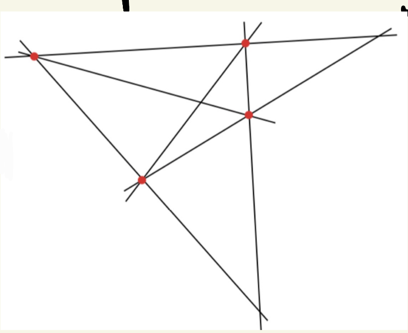
Bi-pyramid reloaded



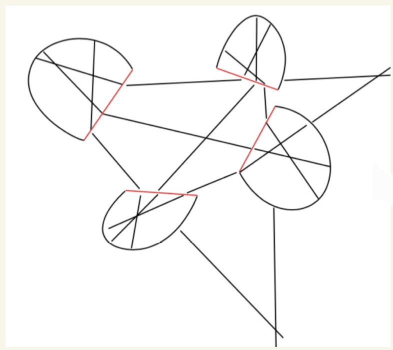
Two of the strata of $\overline{X}_{GP}(3,6)$ $\longrightarrow$ singular point of $\overline{X}(3,6)$

Outside of the planar locus things get complicated ...

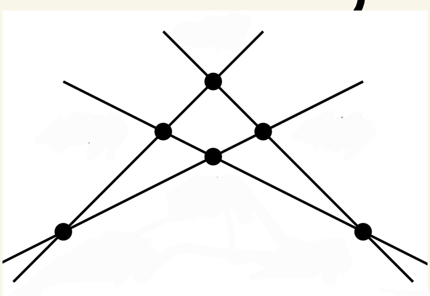
Example A degeneration of six lines



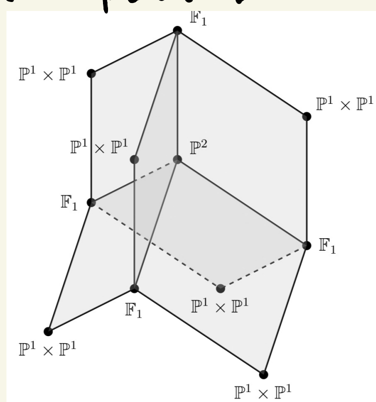
gives a point of $\overline{X}(3,6)$ with a special fiber



The dual degeneration of six points



gives a point of $\overline{X}_{GP}(3,6)$ with dual complex



Thank you for your attention!